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Structural Optimization with FEM-based Shakedown Analyses

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Abstract. In this paper, a mathematical programming formulation is presented for the structural optimization with respect to the shakedown analysis of 3-D perfectly plastic structures on basis of a finite element discretization. A new direct algorithm using plastic sensitivities is employed in solving this optimization formulation. The numerical procedure has been applied to carry out the shakedown analysis of pipe junctions under multi-loading systems. The new approach is compared to so-called *derivative-free* direct search methods. The computational effort of the proposed method is much lower compared to this methods.

Key words: Optimal design, Shakedown analysis, Global optimization, FEM

1. Introduction

In many technically meaningful problems of structures (e.g., in the plant manufacturing) under variable loads the nonlinear behaviour of the material must be considered. Inelastic analyses of the plastic (time-independent) or viscous (time-dependent) behaviour are increasingly used to optimize industrial structures for safety and for an economic operation. Incremental analyses of the path-dependent plastic component behaviour are very time-consuming. The shakedown analysis belongs to the so-called direct or simplified methods which do not achieve the full details of plastic structural behaviour. The objective of shakedown analysis is the determination of an operation regime (i.e., safety margin, load carrying capacities) in which no early failure by plasticity effects has to be expected. Depending on the magnitude of loading, a structure can show the following structural failure modes:

- plastic collapse by unrestricted plastic flow at limit load,
- incremental collapse by accumulation of plastic strains over subsequent load cycles (ratchetting),
- plastic fatigue by alternating plasticity in few load cycles (Low Cycle Fatigue (LCF)),
- plastic instability of slender compression members (not considered in the present paper).

Within the Brite-EuRam Project LISA (Staat and Heitzer, 2001) a procedure is developed using the finite element discretization for direct calculation of the limit and shakedown load of structures made of ductile material. The shakedown analysis is formulated as optimization problem, such that it is easily reformulated for use in a structural optimization process. Determining optimal values for relevant design variables characterizing the geometrical shape as well as the material behaviour requires an efficient strategy to perform sensitivity analyses with respect to the design variables. Heyman (1958) was the first to study the problem of optimal shakedown design (see also (Cohn and Parimi, 1973) for an early contribution). These approaches are restricted to frame structures. Giambanco et al. (1994) extended this approach to circular plates. The presented approach is suitable to general 3-D structures which can be analyzed by a Finite Element code. In the new approach the sensitivity analysis is integrated in the formulation of the shakedown analysis. This permits an integrated treatment of structural and sensitivity analysis and results into easily applicable and efficient numerical algorithms. For a general review of sensitivity methods in nonlinear mechanics see Kleiber et al. (1997). Different types of problems may be considered in structural optimization:

- maximum shakedown load for a given structure (shape).
- optimum shape (e.g. minimum weight) for given shakedown load.

In this contribution we maximize the shakedown range of pipe junctions of variable thickness of the pipe and the junction. Optimal plastic design of frame structures has already been realized in some research level software (Nguyen and Morelle, 1990). Recently, optimal plastic design of plates with holes under multi-loading systems (Schwabe, 2000; Wiechmann et al., 2000) and optimal shakedown design of frame structures (Spiliopoulos, 1999; Tin-Loi, 2000) have been performed.

2. Concepts of shakedown analysis

Static shakedown theorems are formulated in terms of stress and define safe structural states leading to an optimization problem for safe load domains. The maximum safe load domain is the load domain avoiding plastic failure (with the exception of plastic buckling). We restrict our presentation to perfectly plastic material and no elastic failure modes are considered (i.e., no elastic buckling or high cycle fatigue). For a formulation of the shakedown analysis for the more realistic two-surface plasticity material model see, e.g., Heitzer et al. (2000).

2.1. STATIC OR LOWER BOUND SHAKEDOWN ANALYSIS

The shakedown analysis starts from Melan's lower bound theorem for time variant loading for perfectly plastic material. Let us suppose that the loads vary in a convex load domain $\mathcal{L}_0$ such that every load $P(t) = (\mathbf{b}(t), \mathbf{p}(t))$ which lays in $\mathcal{L}_0$ (spanned by NV non-degenerated load vertices P_j), is generated with $\mu_j(t)$ by $P(t) = \mu_1 P_1 + \ldots + \mu_{NV} P_{NV}$. The equilibrium conditions of the shakedown analysis and the yield criterion for the actual stresses have to be fulfilled at every instant of the load history. For the following considerations the von Mises function F is preferred. The maximum enlargement of $\mathcal{L}_0$ is searched for which the structure is safe. The structure is safe against LCF or against ratchetting if there is a stress field $\boldsymbol{\sigma}$(t) such that the equilibrium equations are satisfied and the yield condition (with yield stress σ_y) is nowhere and at no instant t violated.

$$\begin{aligned} \max \quad & \alpha \\ \text{s. t.} \quad & F(\boldsymbol{\sigma}(t)) \leqslant \sigma_y^2 \quad \text{in } V \\ & \text{div}\boldsymbol{\sigma}(t) = -\alpha \mathbf{b}_0(t) \quad \text{in } V \\ & \boldsymbol{\sigma}(t)\,\mathbf{n} = \alpha \mathbf{p}_0(t) \quad \text{on } \partial V_\sigma \end{aligned} \tag{1}$$

for body forces $\alpha\mathbf{b}_0(t)$, surface loads $\alpha\mathbf{p}_0(t)$. By convexity of $\mathcal{L}_0$ the constraints need to be satisfied only in the load vertices P_j. This makes the problem time invariant for any deterministic or stochastic load history.

In the maximum problem (1), the actual stresses $\boldsymbol{\sigma}(t)$ are splitted into fictitious elastic stresses $\boldsymbol{\sigma}^E(t)$ and time independent residual stresses $\boldsymbol{\rho}$

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^E(t) + \boldsymbol{\rho}. \tag{2}$$

All residual stresses $\boldsymbol{\rho}$ generate a linear vector space

$$\mathcal{B} = \{\boldsymbol{\rho} \mid \text{div}\boldsymbol{\rho} = \mathbf{0} \text{ in } V,\, \boldsymbol{\rho}\,\mathbf{n} = 0 \text{ on } \partial V_\sigma\}. \tag{3}$$

Problem (1) can be transformed into a finite optimization problem by FEM discretization. For structures with NG Gaussian points in the FEM model one has to handle $O(NG)$ unknowns and $O(NG)$ constraints. The number of Gaussian points becomes huge for realistic discretizations of industrial structures and no effective solution algorithms for discretizations of the nonlinear optimization problem (1) are available. A method for handling such large-scale optimization problems for perfect plasticity is called *basis reduction technique* or *subspace iteration* (Heitzer, 1999; Heitzer and Staat, 1999; Stein et al., 1993). For a similar formulation of the shakedown problem for the more realistic two-surface plasticity material model using the basis reduction see Heitzer et al. (2000).

With the discretization of the structure in NE finite elements with NG Gaussian points and with the fictitious elastic stresses $\boldsymbol{\sigma}_i^E(j)$ corresponding to the load vertex P_j the following maximum problem has to be solved:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & F(\alpha\boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i) \leqslant \sigma_y^2 \\ & \text{for } i = 1, \ldots, NG,\ j = 1, \ldots, NV,\ \boldsymbol{\rho} \in \mathcal{B} \end{aligned} \tag{4}$$

with the unknowns α and $\boldsymbol{\rho}_i$. Instead of searching in the whole vector space $\mathcal{B}$ for a solution of this problem, a d-dimensional subspace $\mathcal{B}_d^k \subset \mathcal{B}$ is searched for a lower bound factor α^k. Iteratively, different subspaces $\mathcal{B}_d^k$ are chosen in the k-th step of the algorithm for improving the current load factor α^{k-1}. The dimensions of the chosen subspaces are rather small compared to the dimension of $\mathcal{B}$, typically dim $\mathcal{B}_d^k = d \leqslant 6$. The subspaces $\mathcal{B}_d^k$ are generated by an equilibrium iteration with the Finite Element Code PERMAS (Heitzer, 1999; PERMAS, 1988).

This reduction technique generalizes the line search technique, well-known in optimization theory (Fletcher, 1987). Instead of searching the whole feasible region for the optimum a sequence of subspaces with a smaller dimension is chosen and one searches for the best value in these subspaces.

3. Optimization techniques

Hooke and Jeeves coined the phrase *direct search* in a paper that appeared in 1961 (Hooke and Jeeves, 1961). It describes *direct search* by the sequential examination of trial solutions involving comparison of each trial solution with the *best* obtained up to that time together with a strategy for determining (as a function of earlier results) what the next trial solution will be.

To a large extent direct search methods have been replaced by more sophisticated techniques. Many of the direct search methods are based on heuristics which guarantee global convergence behavior analogous to the results known for globalized quasi-Newton techniques. Direct search methods succeed because many of them can be shown to rely on techniques of classical analysis like bisection or golden section search algorithms. For simplicity, we restrict our attention here to unconstrained maximization of function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We assume that f is continuously differentiable, but that information about the gradient of f is either unavailable or unreliable. Because direct search methods neither compute nor approximate derivatives, they are often described as *derivative-free*. For a recent survey on direct search methods and genetic algorithms see Lewis et al. (2000) and Goldberg (1989), respectively. The genetic algorithms are not considered here due to their usually high number of function evaluations; nevertheless they are often useful in structural optimization for elastic material behaviour see, e.g., Woon et al. (2001). A classification of the most methods for numerical optimization can be done according to how many terms of the expansion they need (Lewis et al., 2000), e.g.:

- **Newton's method** (*second order*)
  assumes the availability of first and second derivatives and uses the second-order Taylor polynomial to construct local quadratic approximations of f.
- **Steepest descent** (*first order*)
  assumes the availability of first derivatives and uses the first-order Taylor polynomial to construct local linear approximations of f.

In this classification, *zero-order methods* do not require derivative information and do not construct approximations of f, such that they rely only on values of the objective function. In the following sections the used quasi-Newton methods and pattern search methods are described. For comparison with the proposed new method using plastic sensitivities the optimization code PDS2 (1994) written by Torczon (Lewis et al., 2000) is used.

3.1. QUASI-NEWTON METHODS

A technique used for iteratively solving unconstrained optimization problems is the line search method. The method determines the optimal point on a given line (*search direction*). A back tracking algorithm is used which starts from an initial step length and decreases the step length until it is sufficient (Fletcher, 1987). The algorithm of Dennis/Schnabel is used to omit the exact solution of the one-dimension optimization on the search line (Dennis and Schnabel, 1983).

The IMSL routines BCONF and BCONG are used for the maximization (IMSL, 1997) if analytic gradients are available or not, respectively. The routines BCONF/ BCONG use a quasi-Newton method and an active set strategy to solve maximization problems subject to simple bounds $\mathbf{l}$, $\mathbf{u}$ on the variables. The problem is stated as follows:

$$\begin{aligned} &\max \quad f(\mathbf{x}) \\ &\text{s. t.} \quad \mathbf{l} \leqslant \mathbf{x} \leqslant \mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n \end{aligned} \tag{5}$$

From a given starting point $\mathbf{x}^c$, an active set IA, which contains the indices of the variables at their bounds, is built. A variable is called a *free variable* if it is not in the active set. The routine then computes the search direction for the free variables according to the formula

$$\mathbf{d} = -\mathbf{B}^{-1}\mathbf{g}^c \tag{6}$$

where $\mathbf{B}$ is a positive definite approximation of the Hessian and $\mathbf{g}^c$ is the gradient evaluated at $\mathbf{x}^c$; both are computed with respect to the free variables. Routine BCONF calculates the gradient by a finite-difference method evaluated at $\mathbf{x}^c$. The search direction for the variables in IA is set to zero. A line search is used to find a new point $\mathbf{x}^n$,

$$\mathbf{x}^n = \mathbf{x}^c + \lambda\mathbf{d}, \quad \lambda \in (0, 1] \tag{7}$$

such that

$$f(\mathbf{x}^n) \leqslant f(\mathbf{x}^c) + \alpha\mathbf{g}^T\mathbf{d}, \alpha \in (0, 0.5) \tag{8}$$

Finally, the optimality conditions

$$||g(\mathbf{x}_i)|| \leqslant \epsilon, l_i < x_i < u_i \tag{9}$$

$$g(\mathbf{x}_i) < 0, x_i = u_i \tag{10}$$

$$g(\mathbf{x}_i) > 0, x_i = l_i \tag{11}$$

are checked, where ϵ is a gradient tolerance. When optimality is not achieved, $\mathbf{B}$ is updated according to the BFGS formula (e.g., Fletcher, 1987):

$$\mathbf{B} \leftarrow \mathbf{B} - \frac{\mathbf{B}\mathbf{s}\mathbf{s}^T\mathbf{B}}{\mathbf{s}^T\mathbf{B}\mathbf{s}} + \frac{\mathbf{y}\mathbf{y}^T}{\mathbf{y}^T\mathbf{s}} \tag{12}$$

where $\mathbf{s} = \mathbf{x}^n - \mathbf{x}^c$ and $\mathbf{y} = \mathbf{g}^n - \mathbf{g}^c$. Another search direction is then computed to begin the next iteration.

The active set is changed only when a free variable hits its bounds during an iteration or the optimality condition is met for the free variables but not for all variables in IA, the active set. In the latter case, a variable that violates the optimality condition will be dropped out of IA. For more details on the quasi-Newton method and line search, see Dennis and Schnabel (1983). Since a finite-difference method is used in routine BCONF to estimate the gradient for some single precision calculations, an inaccurate estimate of the gradient may cause the algorithm to terminate at a noncritical point.

3.2. PATTERN SEARCH METHOD

Pattern search methods are characterized by a series of *exploratory moves* that consider the behavior of the objective function at a pattern of points, all of which lie on a rational lattice. The exploratory moves consist of a systematic strategy for visiting the points in the lattice in the immediate vicinity of the current iterate (Lewis et al., 2000). For each move the parameter is varied and it is decided if there is an improvement, such that the procedure is a direct search. Recently, a general theory for pattern search (Torczon, 1997) extended a global convergence analysis (Torczon, 1991) of the multi-directional search algorithm. The used multi-directional search algorithm proceeds by reflecting a simplex through the centroid of one of the faces (Lewis et al., 2000).

A simplex is a set of n+1 points in $\mathbb{R}^n$, e.g. a triangle in $\mathbb{R}^2$ and a tetrahedron in $\mathbb{R}^3$, etc. A non-degenerate simplex is one for which the set of edges adjacent to any vertex in the simplex forms a basis for the space. If one replaces a vertex by reflecting it through the centroid of the opposite face, then the result is also a simplex (see Figure 1). The first single move is that of reflection which identifies the *worst* vertex in the simplex (i.e., the one with the least desirable objective value) and then reflects the worst simplex through the centroid of the opposite face. If the reflected vertex is still the worst vertex, then next choose the *next worst* vertex and repeat the process. The ultimate goals are either to replace the *best* vertex or to ascertain that the best vertex is a candidate for a maximizer. Until then, the algorithm keeps moving the simplex by flipping some vertex (other than the best vertex) through the centroid of the opposite face. An *expansion step* allows for a more progressive move by doubling the length of the step from the centroid to the reflection point, whereas a *contraction steps* allow for more conservative moves by halving the length of the step from the centroid to either the reflection point or the worst vertex. These steps allow a deformation of the shape of the original simplex.

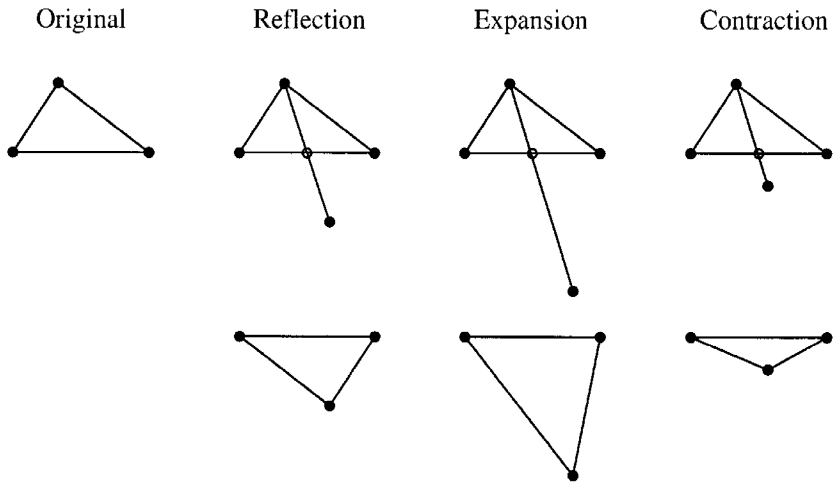

Figure 1. Simplex with the reflection of one vertex through the centroid of the opposite face.

4. Sensitivity

The line search method needs for solving optimization problems a search direction. Here the search line is given by the sensitivities of the shakedown analysis, i.e., the search line is given by the gradients of the shakedown factor with respect to the design parameters. The following sections describe how to obtain these gradients directly from the shakedown analysis.

4.1. SENSITIVITY AND MATHEMATICAL PROGRAMMING

A constraint maximization problem **P** in the most general case is defined as

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s. t.} \quad & g_i(\mathbf{x}) \leqslant 0, \forall i \in \mathcal{I}. \end{aligned} \tag{13}$$

Suppose that $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable and let $\mathcal{I}$ be some index set. In many applications (e.g., shakedown analysis), the objective function f as well as the constraint functions g_i may depend also on other parameters. Consider the following perturbation $\mathbf{P}(\boldsymbol{\varepsilon})$ of the original problem $\mathbf{P}(\mathbf{0})$

$$\begin{aligned} \max \quad & f(\mathbf{x}, \boldsymbol{\varepsilon}) \\ \text{s. t.} \quad & g_i(\mathbf{x}, \boldsymbol{\varepsilon}) \leqslant 0, \forall i \in \mathcal{I}, \boldsymbol{\varepsilon} \in \mathbb{R}^q, q \in \mathbb{N} \end{aligned} \tag{14}$$

A perturbation $\boldsymbol{\varepsilon}$ can be interpreted in two ways: as a *random* error, or as a *specific* change in the parameters defining the problem functions. The optimal solution $\mathbf{x}^*(\boldsymbol{\varepsilon})$ of problem $\mathbf{P}(\boldsymbol{\varepsilon})$ with the Lagrangian multipliers $\boldsymbol{\lambda}^*$ fulfills the following first order Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned}&\lambda_i^* g_i(\mathbf{x}^*, \boldsymbol{\varepsilon}) = 0, \forall i \in \mathcal{I}\\&\lambda_i^* \geqslant 0, \forall i \in \mathcal{I}\\&\nabla_x L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\varepsilon}) = \mathbf{0}\end{aligned} \tag{15}$$

with the Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = f(\mathbf{x}, \boldsymbol{\varepsilon}) - \sum_{i \in \mathcal{I}} \lambda_i g_i(\mathbf{x}, \boldsymbol{\varepsilon}). \tag{16}$$

If the system of equations is nonsingular, the implicit function theorem implies, the existence of a unique differentiable local solution $(\mathbf{x}^*(\boldsymbol{\varepsilon}), \boldsymbol{\lambda}^*(\boldsymbol{\varepsilon}))$ of $\mathbf{P}(\boldsymbol{\varepsilon})$. For second order KKT conditions the restricted Lagrangian is defined with only the active constraints $g_i = 0, i \in \mathcal{I}_0$. The second-order sufficient conditions state that a point $(\mathbf{x}^*, \boldsymbol{\varepsilon}^*)$ is a strict local maximum of $\mathbf{P}(\boldsymbol{\varepsilon})$ if (15) is satisfied at $(\mathbf{x}^*, \boldsymbol{\varepsilon})$ and if the Hessian $\nabla_x^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\varepsilon})$ of the restricted Lagrangian is negative definite on the tangent space $\{\boldsymbol{\xi} \mid \boldsymbol{\xi}^T \nabla_x g_i(\mathbf{x}^*) = 0,\ i \in \mathcal{I}_0 : \lambda_i^* > 0\}$. Let $\boldsymbol{\varepsilon} = \mathbf{0}$, then the conditions are fulfilled in a local solution $\mathbf{x}^*$ of $\mathbf{P}(\mathbf{0})$. The associated theorem is given in Fiacco (1976, 1983).

COROLLARY. (Fiacco, 1976): *At a local solution* $\mathbf{x}^*$ *of problem* $\mathbf{P}(\mathbf{0})$*, assume that the linear independence condition, the second order sufficiency condition and the strict complementarity condition* $\lambda_i^* g_i(\mathbf{x}^*, \boldsymbol{\varepsilon}) = 0$ *are satisfied for all* $i \in \mathcal{I}$*, and that the functions defining* $\mathbf{P}(\boldsymbol{\varepsilon})$ *are twice continuously differentiable with respect to* $(\mathbf{x}, \boldsymbol{\varepsilon})$ *in a neighbourhood of* $(\mathbf{x}^*, \mathbf{0})$*. It follows that at,* $\boldsymbol{\varepsilon}_0 = \mathbf{0}$

$$\frac{d}{d\boldsymbol{\varepsilon}} \begin{pmatrix} \mathbf{x}(\mathbf{0}) \\ \boldsymbol{\lambda}(\mathbf{0}) \end{pmatrix} = -Q_0^{-1} V_0, \tag{17}$$

and

$$\frac{d}{d\boldsymbol{\varepsilon}} f(\mathbf{x}(\mathbf{0}), \mathbf{0}) = \frac{\partial f(\mathbf{x}(\mathbf{0}), \mathbf{0})}{\partial \boldsymbol{\varepsilon}} - \sum_{i \in \mathcal{I}} \lambda_i^* \frac{\partial g_i(\mathbf{x}(\mathbf{0}), \mathbf{0})}{\partial \boldsymbol{\varepsilon}} \tag{18}$$

where

$$Q_0 = \begin{pmatrix} \nabla_x^2 L & -\nabla_x g_1 & \cdots & -\nabla_x g_m \\ \lambda_1 \nabla_x^T g_1 & g_1 & & 0 \\ \vdots & & \ddots & \\ \lambda_m \nabla_x^T g_m & 0 & & g_m \end{pmatrix} \tag{19}$$

and

$$V_0 = \begin{pmatrix} \frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[\nabla_x L^T\right] \\ \lambda_1 \frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[\nabla_x g_1\right] \\ \vdots \\ \lambda_m \frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[\nabla_x g_m\right] \end{pmatrix}. \tag{20}$$

All quantities are evaluated at $\mathbf{x}^*(\mathbf{0})$, $\boldsymbol{\lambda}^*(\mathbf{0})$, $\boldsymbol{\varepsilon}_0$ *with* $m = |\mathcal{I}|$.

4.2. SENSITIVITY IN SHAKEDOWN ANALYSIS

We restrict ourselves to the following perturbation $\mathbf{P}(\boldsymbol{\varepsilon})$ of the original shakedown problem with the unknowns $\mathbf{x} = (\alpha, \boldsymbol{\rho}_1, \dots \boldsymbol{\rho}_{NG})$

$$\begin{aligned} &\max \quad \alpha \\ &\text{s. t.} \quad F[\alpha\boldsymbol{\sigma}^E_{i,j}(\boldsymbol{\varepsilon}) + \boldsymbol{\rho}_i] - \sigma_y^2 \leqslant 0, \quad i = 1, \dots, NG, \; j = 1, \dots, NV. \end{aligned} \tag{21}$$

The problem fulfills the assumptions of the corollary, such that we obtain the derivatives at the solution α^* of the original problem $\mathbf{P}(\mathbf{0})$ with the original fictitious elastic stresses $\boldsymbol{\sigma}^E(\mathbf{0})$ by:

$$\begin{aligned} \frac{d}{d\boldsymbol{\varepsilon}} f(\mathbf{x}(\mathbf{0}), \mathbf{0}) &= \frac{d\alpha^*}{d\boldsymbol{\varepsilon}} - \sum_{i \text{ active}} \lambda_i^* \frac{\partial g_i(\mathbf{x}(\mathbf{0}), \mathbf{0})}{\partial \boldsymbol{\varepsilon}} \Bigg|_{\varepsilon=0} \\ &= -\alpha^* \sum_{i \text{ active}} \lambda_i^* \frac{\partial}{\partial \boldsymbol{\sigma}_i^E} \left[F(\alpha\boldsymbol{\sigma}_i^E(\boldsymbol{\varepsilon}) + \boldsymbol{\rho}_i) \right] \frac{\partial \boldsymbol{\sigma}_i^E(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \Bigg|_{\varepsilon=0} \end{aligned} \tag{22}$$

This problem is solved by the described basis reduction method in a recursive manner by means of Sequential Quadratic Programming (SQP) techniques. The shakedown factor α_k as well as the Lagrange multipliers λ_k^* obtained during the optimization step k converge to the true solution α^* and $\boldsymbol{\lambda}^*$ (Heitzer, 1999). Therefore, in Eq. (22) all values except

$$\frac{\partial \boldsymbol{\sigma}_i^E(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \Bigg|_{\varepsilon_0} \tag{23}$$

are given by the shakedown analysis. This means, that in the case of shakedown analysis the sensitivity analysis of the plastic structural behaviour is reducible to the sensitivity analysis of the elastic structural response, which is a significant reduction of computational effort. The sensitivity analysis of the elastic response is performed by a finite-difference method for a small number of parameters, see Kleiber et al. (1997) for alternative techniques.

5. Pipe-junction subjected to internal pressure and temperature loads

A pipe-junction subjected to internal pressure and temperature loads is analyzed as a simple example. The shakedown analyses are performed for perfectly plastic material with a yield stress $\sigma_y = 250$ N/mm^2. The inner diameters $D = 39$ mm and $d = 15$ mm of the pipe and of the junction are fixed, respectively. The length of the pipe and of the junction are $L = 81.9$ mm and $l = 17.1$ mm fixed, respectively. The variable dimensions are the wall-thickness s and t of the pipe and the junction, respectively. The meshes of the pipe-junction are generated my an automatic mesh-generator. The different models are discretized with 125 solid 20-node hexahedron elements (HEXEC20). The dimensions of the model are based on a pipe benchmark problem of PERMAS (PERMAS, 1988). The FE-mesh and the essential dimensions of the different pipe-junctions are represented in Figure 2.

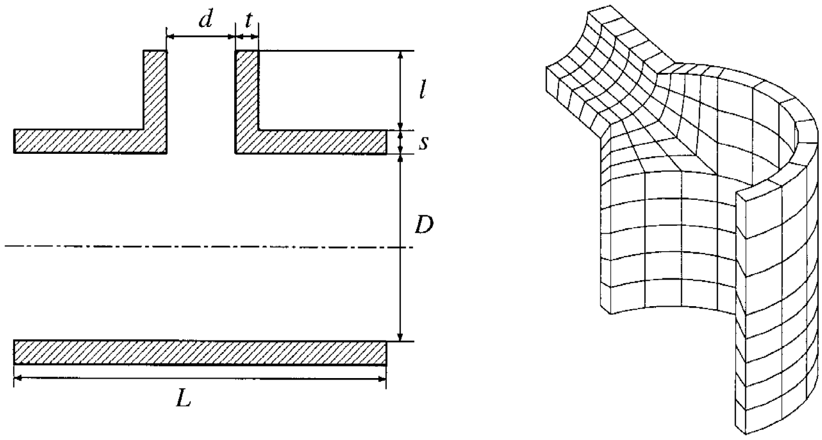

Figure 2. FE-mesh and dimensions of the pipe-junction.

The pipe junction is subjected to two-parameter loading, i.e. pressure P and temperature $T = T_o - T_i$ with T_o outer and T_i inner temperature vary independentally

$$\begin{aligned} &0 \leqslant P \leqslant \alpha\mu_1 P_0, \\ &0 \leqslant T_i \leqslant \alpha\mu_2 T_0, \qquad 0 \leqslant \mu_1, \mu_2 \leqslant 1. \end{aligned}$$

P_0 and T_0 are a reference pressure and temperature difference, respectively.

The goal of the structural optimization in this example is to maximize the shakedown factor α for the wall-thickness s and t varying in given bounds:

$$\begin{aligned} \max \quad & \alpha_s \\ \text{s. t.} \quad & 0 < s, t \leqslant 7.5. \end{aligned} \tag{24}$$

For pure pressure variation the optimal wall thickness of the pipe and of the junction will tend to infinity because of the decreasing elastic stresses. On the other hand for pure temperature variation the wall thickness of the pipe and of the junction will tend to zero. Therefore, different finite positive ratios between the initial pressure P_0 and the initial temperature T_0 are chosen. The design variables vary between bounds $0 < s, t \leqslant 7.5$ mm to guarantee that the chosen mesh of the pipe-junction is not degenerated, otherwise the chosen automatic mesh-generator leads to meshes with degenerated elements. The shakedown factor α_s is the solution of the corresponding shakedown optimization problem with the load domain $\mathcal{L}_0$ defined by P_0 and T_0.

Three different mathematical optimization codes using the FEM-based shakedown analysis are compared for this example. The direct search algorithm PDS2 and the IMSL routines BCONF and BCONG, using finite-difference method and analytical gradients, respectively. All codes use the same subroutine to calculate the shakedown load factor α_s. The direct search algorithm uses a fixed search pattern. BCONF performs a finite-difference method to estimate the gradient, such that for each gradient at least two additional shakedown analyses have to be performed. BCONG uses the given analytical gradients. The sensitivities are calculated using a finite-difference method for the elastic stresses and the algorithm described above.

A comparison of the different methods is shown in Table 1. In addition to the optimal values s^*, t^* and the corresponding shakedown load magnitudes $P_s = \alpha_s P_0$ and $T_s = \alpha_s T_0$ the number of function calls (i.e., shakedown analyses) are given. The results for the methods are comparatively close. The values s^*, t^* decrease for increasing temperatures as expected. The direct search algorithm is quite sensitive to the initial starting points, because of its fixed search scheme. Nevertheless with improved starting values the results for the pattern search method correspond well with the BCONG routine using the new implemented sensitivities. In all cases the new method is the fastest method in terms of function calls. It is evident, that the use of the analytical gradients (BCONG) is preferable to the use of the finite-difference gradients (BCONF). It has to be noticed that the function $f(s, t) = \alpha_s$ is not convex, such that probably local maxima exist in the region $0 < s, t \leqslant 7.5$ mm. For instance in load level $T_i^0/P^0 = 100$ [K/MPa] the resulting shakedown load magnitudes P_s and T_s are close, whereas the values s^*, t^* are fairly different. This may indicate that the global maximum in the region $0 < s, t \leqslant 7.5$ mm has not yet reached. Additional computations suggest, that the temperature difference $T_s \approx 1400$ K is the highest allowable temperature load in the region $0 < s, t \leqslant 7.5$ mm.

6. Conclusion

Shakedown theorems are exact theories of classical plasticity for the direct computation of safety factors or of the load carrying capacity under varying loads. This method can be based on static and kinematic theorems for lower and upper bound analysis. Using Finite Element Methods more realistic modeling can be used for a more rational design. A mathematical programming formulation is presented for the structural optimization with respect to the shakedown analysis of 3-D perfectly plastic structures. A new direct algorithm using plastic sensitivities is employed in solving this optimization formulation. The numerical procedure has been applied to carry out the shakedown analysis of pipe junctions under multi-loading systems. The computational effort of the proposed method is much lower compared to so-called *derivative-free* direct search methods.

Table 1. Comparison of the different methods

T_i^0 / P^0 (K/MPa)	Method	s^* (mm)	t^* (mm)	P_s (MPa)	T_s (K)	Function calls
20	Direct	7.3542	4.6692	27.06	541	37
	BCONF	7.5000	6.3031	29.15	583	64
	BCONG	7.5000	4.4011	26.35	527	7
40	Direct	7.4665	6.8200	19.90	796	17
	BCONF	5.5654	3.7504	18.08	720	52
	BCONG	7.5000	6.3041	19.75	790	12
60	Direct	7.3994	4.8968	15.51	931	18
	BCONF	6.7809	4.5379	15.37	922	45
	BCONG	4.7345	7.5000	14.62	877	13
80	Direct	6.9828	6.6913	13.52	1082	50
	BCONF	4.1465	5.7304	11.65	932	45
	BCONG	7.1101	4.9480	13.15	1052	4
100	Direct	5.1344	4.0902	10.07	1007	22
	BCONF	4.5733	3.4209	10.71	1072	41
	BCONG	4.1404	4.2517	10.10	1011	9
120	Direct	6.9835	6.6906	10.35	1242	7
	BCONF	7.5000	4.4788	9.13	1096	27
	BCONG	7.5000	6.7115	10.57	1268	6
140	Direct	5.8310	4.3171	8.13	1138	17
	BCONF	5.6060	3.4642	7.73	1082	44
	BCONG	4.4660	4.3343	8.51	1191	4
160	Direct	5.2488	4.6217	7.21	1154	41
	BCONF	4.9742	4.0640	7.81	1249	75
	BCONG	3.8007	2.9375	6.96	1114	12
180	Direct	6.1470	6.9148	7.31	1326	14
	BCONF	4.4983	4.5002	6.28	1130	21
	BCONG	5.4341	3.7255	6.69	1205	7
200	Direct	6.0650	5.7721	6.60	1320	15
	BCONF	4.4038	3.9833	6.43	1286	68
	BCONG	3.9576	3.5771	6.03	1207	5

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